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NAVORD REPORT 4022

THE LAMINAR BOUNDARY LAYER ON A ROTATING CYLINDER IN CROSSFLOW

20 JUNE 1955

U. S. NAVAL ORDNANCE LABORATORY
WHITE OAK, MARYLAND

NAVORD Report 4022

Aeroballistic Research Report 288

THE LAMINAR BOUNDARY LAYER ON A ROTATING CYLINDER IN CROSSFLOW

Prepared by:

E. Krahn

ABSTRACT: A rotating cylinder in a stream produces a circulation which is regarded as the cause of the Magnus force. The problem is to find the dependence of the circulation on the rotational speed of the cylinder. This question is treated in the present report for a stationary flow about a circular cylinder with the axis perpendicular to the direction of the stream under the assumption that the flow in the boundary layer is laminar. Two approximate methods are used for the calculation of the boundary layer. One is due to Burgers and the other is an adaptation of the Polhausen method. In the case of only one stagnation line on the surface of the cylinder the boundary layer is evaluated numerically, the profiles and the shearing stress computed.

U. S. NAVAL ORDNANCE LABORATORY
WHITE OAK, SILVER SPRING, MARYLAND

20 June 1955

This report contains information concerning the laminar boundary layer on a rotating cylinder and the circulation in the potential flow around the cylinder. It is a step in the calculation of the Magnus force.

This work has been carried out at the Naval Ordnance Laboratory under Task NOL - A3d - 453 - 1 - 55.

The results are distributed to outside research agencies for information and for use in the study of spinning bodies of revolution.

JOHN T. HAYWARD
Captain, USN
Commander

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Aeroballistic Research Department
By direction

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ILLUSTRATIONS

THE LAMINAR BOUNDARY LAYER ON A ROTATING CYLINDER IN CROSSFLOW

1. An infinite cylinder is rotating in a stream perpendicular to the axis and a stationary flow has developed. The fluid is supposed to be incompressible but viscous. The Reynolds number is assumed small enough to insure a laminar boundary layer. The question is, how does the circulation in the surrounding flow depend on the speed of the rotation? The answer will be sought by the calculation of the boundary layer. This will be carried out by two different methods independently. First by an approximate method due to J. M. Burgers and then by the solution of the momentum equation.

2. J. M. Burgers [1] calculated the flow around a non-rotating cylinder by a method which has not been much used later. He got a separation of the flow at an angle of 120° from the stagnation point, a result which does not agree with the experiments nor with other calculations. This is due to the fact that he took the pressure distribution of the surrounding potential flow as given in the case of a stream attached to the cylinder on the whole surface. Hiemenz [2] carried out the calculation of the same flow by a different method by using a velocity distribution of the potential flow taken from experiments and got a separation at 83°. Applying the method of Blasius one gets the same result, which coincides with the experimental data.

Recalculating Burgers' result with the pressure distribution used by Hiemenz one gets the point of separation at 88°, which does not differ so very much from the exact result. It is therefore of interest to use this method in the calculation of the boundary layer on the rotating cylinder, so much the more as it is easier to apply than other methods.

3. Notations.

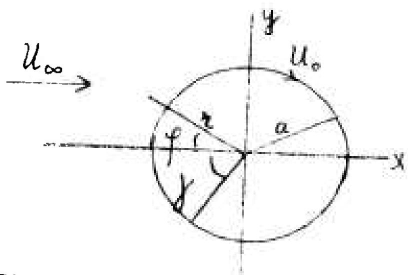


Fig. 1

x, y — Cartesian coordinates
r, φ — polar coordinates
a — radius of the cylinder
l — length of the cylinder
U_o — velocity of the surface of the cylinder
U_∞ — velocity of the potential flow at infinity
U — velocity of the potential flow near the surface of the cylinder

u — velocity component of the flow in the boundary layer in the direction parallel to the surface of the cylinder.

v — velocity component of the flow in the boundary layer in the r-direction

ω — vorticity

ϕ, ψ — potential and stream function of the potential flow

v_x, v_y — velocity components in the boundary layer parallel to the x– and the y–axis.

γ — angle between the radius to the stagnation point and the negative x – axis

ρ — density

μ — viscosity

τ — shearing stress

ν — kinematic viscosity

Re — Reynolds' number

Γ — circulation

δ — thickness of the boundary layer

η — distance from the cylinder in radial direction

K — form parameter

$X = \frac{U}{U_o}$

$Y = \frac{KU}{10U_o}$

$c = \frac{4U_\infty}{U_o}$

ξ — parameter

$t = \frac{\eta}{\delta}$

P — power

4. The stationary potential flow around the cylinder is given by

$$\Phi = -U_\infty\left(r + \frac{a^2}{r}\right)\cos\varphi + \frac{\Gamma}{2\pi}\cdot\varphi$$

$$\Psi = U_\infty\left(r - \frac{a^2}{r}\right)\sin\varphi + \frac{\Gamma}{2\pi}\log\frac{r}{a}$$

$$U = U_\infty\left(1 + \frac{a^2}{r^2}\right)\sin\varphi + \frac{\Gamma}{2\pi}\cdot\frac{1}{r}\,.$$

If the circulation Γ does not exceed $4\pi U_\infty a$ then there are one or two stagnation points on the circle (stagnation lines on the cylinder) and Γ can be written as

$$\Gamma = 4\pi U_\infty a \sin\gamma .$$

From now on, this expression for Γ (which is a restriction on the amount of circulation) will be used.

For the flow near the surface of the cylinder one can take $r = a$ and has then

$$U = 2 U_\infty (\sin\varphi + \sin\gamma)$$

$$\phi = 2 U_\infty a (\varphi \sin\gamma - \cos\varphi) .$$

These expressions will be further used for U and ϕ.

5. Burgers' method consists in starting from the equation of the vorticity in a stationary flow

$$v_x \frac{\partial\omega}{\partial x} + v_y \frac{\partial\omega}{\partial y} = \nu\left(\frac{\partial^2\omega}{\partial x^2} + \frac{\partial^2\omega}{\partial y^2}\right) \tag{1}$$

and substituting for the velocity components in the boundary layer those of the potential flow outside the boundary layer, i.e.

$$v_x = \frac{\partial\psi}{\partial y} \qquad v_y = -\frac{\partial\psi}{\partial x} \quad ,$$

where ψ is the stream function of the potential flow. Transforming (1) from the variables x,y to the variables ϕ,ψ he gets

$$\frac{\partial\omega}{\partial\phi} = \nu\left(\frac{\partial^2\omega}{\partial\phi^2} + \frac{\partial^2\omega}{\partial\psi^2}\right) .$$

In conformity with usual boundary layer simplifications $\frac{\partial^2\omega}{\partial\phi^2}$ is dropped as small compared with $\frac{\partial^2\omega}{\partial\psi^2}$ and

$$\frac{\partial\omega}{\partial\phi} = \nu \frac{\partial^2\omega}{\partial\psi^2} \tag{2}$$

results. A solution of (2) is

$$\omega = \frac{1}{\sqrt{\phi - \xi}} \cdot e^{-\frac{\psi^2}{4\nu(\phi-\xi)}} ,$$

where ξ is an arbitrary parameter.

As (2) is a linear equation, then with an arbitrary function A(ξ)

$$\omega = -\frac{1}{2}\int_{\phi_1}^{\phi} \frac{A(\xi)}{\sqrt{\pi\nu(\phi-\xi)}} \cdot e^{-\frac{\Psi^2}{4\nu(\phi-\xi)}} d\xi$$

is also a solution. ϕ_1 is the value of ϕ at the forward stagnation point.

To get the velocity component u in the boundary layer in the direction tangential to the circle (surface of the cylinder) $\frac{\partial v}{\partial \varphi}$ is neglected compared with $\frac{\partial u}{\partial r}$ in

$$\omega = \frac{1}{2}\left(\frac{\partial v}{r\partial \varphi} - \frac{\partial u}{\partial r}\right)$$

as v, the velocity component in the radial direction, is zero at the surface and small inside the boundary layer. Then

$$\omega = -\frac{1}{2}\frac{\partial u}{\partial r} = -\frac{1}{2}\frac{\partial u}{\partial \Psi}\cdot U .$$

The application of this method to the problem at hand requires now the determination of the constants and the function A(ξ).

For $\Psi = 0$ there must be $u = U_0$ and for sufficiently large Ψ there must be $u = U$.
Writing

$$u = U_0 + \frac{1}{U}\int_{\phi_1}^{\phi} A(\xi)\left\{\int_0^{\Psi} \frac{e^{-\frac{\Psi^2}{4\nu(\phi-\xi)}}}{\sqrt{\pi\nu(\phi-\xi)}} d\Psi\right\} d\xi$$

and taking into account

$$\int_0^{\infty} \frac{e^{-\frac{\Psi^2}{4\nu(\phi-\xi)}}}{\sqrt{\pi\nu(\phi-\xi)}} d\Psi = 1 ,$$

the condition for A(ξ) becomes

$$U = U_0 + \frac{1}{U}\int_{\phi_1}^{\phi} A(\xi)\, d\xi .$$

Thus

$$A(\phi) = (2U - U_0)\frac{dU}{d\phi}$$

and

$$(3) \qquad u = U_0 + \frac{1}{U}\int_{\phi_1}^{\phi} (2U-U_0)\frac{dU}{d\xi}\left\{\int_0^{\Psi} \frac{e^{-\frac{\Psi^2}{4\nu(\phi-\xi)}}}{\sqrt{\pi\nu(\phi-\xi)}} d\Psi\right\} d\xi .$$

Here ϕ a ı ψ of the potential flow are regarded as coordinates in the boundary ayer, where ψ is not the stream function of the viscous flow.

U is regarded as a function of ϕ only. This corresponds to the assumption of a pressure in the boundary layer, which depends on φ only and not on r.

Velocity profiles can be easily computed by (3), but one should remember that (3) is an approximate expression only, representing u correctly at the surface of the cylinder and at the outer limit of the boundary layer.

6. To determine the circulation about the cylinder from (3) small values of ψ will be considered and higher than first order terms in ψ will be neglected. Then

$$\int_0^{\Psi} \frac{e^{-\frac{\psi^2}{4\nu(\phi-\xi)}}}{\sqrt{\pi\nu(\phi-\xi)}}\, d\Psi = \frac{\Psi}{\sqrt{\pi\nu(\phi-\xi)}}\,.$$

Making the substitution

$$r = a + \eta$$

and retaining linear terms only in η

$$\Psi = 2\,U_\infty \eta (\sin\varphi + \sin\gamma) = U\eta\,.$$

(3) becomes thus

$$u = U_0 + \frac{\eta}{U}\int_{\phi_1}^{\phi} (2U - U_0)\frac{dU}{d\xi}\cdot\frac{U}{\sqrt{\pi\nu(\phi-\xi)}}\, d\xi\,.$$

The integral must be equal to zero for $U = 0$ to give a finite value to u. That is the case at the forward stagnation point where $\phi = \phi_1$. At the rear stagnation point it gives a condition for U_o. Inserting the values of U and ϕ

$$\int_{-\gamma}^{\pi+\gamma} \frac{\left(\sin\varphi + \sin\gamma - \frac{U_0}{4U_\infty}\right)\cos\varphi\,(\sin\varphi + \sin\gamma)}{\sqrt{(\pi+\gamma)\sin\gamma - \cos(\pi+\gamma) - \varphi\sin\gamma + \cos\varphi}}\, d\varphi = 0$$

or

$$\text{(4)}\quad \frac{U_0}{4U_\infty} = \int_{-\gamma}^{\pi+\gamma} \frac{(\sin\varphi + \sin\gamma)^2\cos\varphi}{\sqrt{\cos\varphi + \cos\gamma + (\pi+\gamma-\varphi)\sin\gamma}}\, d\varphi : \int_{-\gamma}^{\pi+\gamma} \frac{(\sin\varphi + \sin\gamma)\cos\varphi}{\sqrt{\cos\varphi + \cos\gamma + (\pi+\gamma-\varphi)\sin\gamma}}\, d\varphi\,.$$

The denominator of each integral is zero at $\varphi = \pi + \gamma$, nevertheless the integrals converge. This can be seen if ϵ is introduced by the substitution

$$\varphi = \pi + \gamma - \epsilon$$

and the integrals calculated from $x+\gamma-\epsilon$ to $x+\gamma$. They become then

$$-\cos^3\gamma\sqrt{\frac{2}{\cos\gamma}}\cdot\frac{\epsilon^2}{2} \quad \text{and} \quad -\cos^2\gamma\sqrt{\frac{2}{\cos\gamma}}\cdot\epsilon$$

respectively. This proves the statement.

The integrals in (4) were evaluated numerically, except in the case $\gamma = 0$ where an exact integration is possible, and the following values were found

γ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\frac{U_0}{U_\infty}$	1.6	2.5	3.2	3.6

These results need some explanation.

If the cylinder does not rotate then the separation occurs on both sides, let us call them the upper and the lower sides, symmetrically. If the cylinder rotates in such a way that on the upper side the rotation is in the direction of the flow, then on this side the point of separation will be farther from the forward stagnation point than in the case of the non-rotating cylinder, and on the lower side it will be nearer.

The result for $\gamma = 0$ means that in case of a speed ratio 1.6 the flow will be attached to the cylinder on the upper half, whereas on the lower side it separates and therefore the velocity distribution assumed for U does not apply here.

Similarly in case of $\gamma = \frac{\pi}{6}$ and $\gamma = \frac{\pi}{3}$, where the calculation is valid on the upper part from stagnation point to stagnation point.

One cannot expect from these results that they are precise, because in deriving (4) too many terms were neglected. One should not be surprised if the figures are erroneous by 10 or 20 per cent. Nevertheless they are of the same order as the experimental results and show that e.g. in the case of $\gamma = \frac{\pi}{2}$ the circulation in the boundary layer decreases from its value at the surface of the cylinder to about half its amount in the potential flow, for at the surface

$$\Gamma = 2\pi a U_0 = 7.2\pi a U_\infty$$

whereas in the potential flow

$$\Gamma = 4\pi a U_\infty .$$

7. To get more reliable results a different method will be applied. The boundary layer equation shall be taken in the form

$$u\frac{\partial u}{a\partial\varphi} + v\frac{\partial u}{\partial z} = U\frac{dU}{a\,d\varphi} + \nu\frac{\partial^2 u}{\partial z^2} . \tag{5}$$

Here the centrifugal term is omitted. This is justifiable as long as the thickness of the boundary layer is small compared with the radius of the cylinder.

The other equation is the continuity equation

(6) $$\frac{1}{a}\frac{\partial u}{\partial \varphi} + \frac{\partial v}{\partial r} = 0.$$

The boundary conditions are

(7) $$\begin{array}{llll} \text{for } r = a: & u = U_0 & v = 0 & \\ \text{for } r = a+\delta: & u = U & \frac{\partial u}{\partial r} = 0 & \frac{\partial^2 u}{\partial r^2} = 0, \end{array}$$

where δ is the boundary layer thickness.

Further the variable

$$\eta = r - a$$

will be used.

Eliminating v from (5) and (6) and integrating over the thickness of the boundary layer the momentum equation results

(8) $$2\int_0^\delta u \frac{\partial u}{\partial \varphi} d\eta - U\int_0^\delta \frac{\partial u}{\partial \varphi} d\eta = U \frac{dU}{d\varphi}\cdot\delta - a\nu \frac{\partial u}{\partial \eta}\Big|_{\eta=0} .$$

8. The solution will be sought in the form

$$u = \sum_{k=0}^{4} a_k t^k ,$$

where the a_k are functions of φ and $t = \frac{\eta}{\delta}$.

By the 5 boundary conditions (7) the 5 functions a_k are determined and the solution is

(9) $$u = U_0 + (2t - 2t^3 + t^4)(U - U_0) + \frac{1}{6}(t - 3t^2 + 3t^3 - t^4) K U ,$$

where

(10) $$K = \frac{\delta^2}{a\nu} \cdot \frac{dU}{d\varphi} .$$

Only the boundary layer thickness δ will be used, as the displacement thickness

$$\int_0^\delta \left(1 - \frac{u}{U}\right) d\eta$$

does not seem reasonable in the present case. It could be replaced perhaps by

$$\int_0^\delta \frac{u - U}{U_0 - U} d\eta \quad \text{or} \quad \int_0^\delta \frac{u - U}{U_0} d\eta .$$

9. Substituting (9) in (8) one gets the following equation

$$2\delta\left\{\frac{7}{10}U_0\frac{dU}{d\varphi}+\frac{1}{120}U_0\frac{d(KU)}{d\varphi}+\frac{367}{630}(U-U_0)\frac{dU}{d\varphi}+\frac{71}{15120}(U-U_0)\frac{d(KU)}{d\varphi}+\right.$$

$$\left.+\frac{71}{15120}KU\frac{dU}{d\varphi}+\frac{1}{9072}KU\frac{d(KU)}{d\varphi}\right\}+$$

$$(11)\quad +2\frac{d\delta}{d\varphi}\left\{-\frac{3}{10}U_0(U-U_0)+\frac{1}{120}U_0KU-\frac{263}{1260}(U-U_0)^2+\frac{71}{15120}(U-U_0)KU+\right.$$

$$\left.+\frac{1}{18144}(KU)^2\right\}-$$

$$-U\delta\left\{\frac{7}{10}\frac{dU}{d\varphi}+\frac{1}{120}\frac{d(KU)}{d\varphi}\right\}+\frac{d\delta}{d\varphi}\left\{\frac{3}{10}U(U-U_0)-\frac{1}{120}U.KU\right\}=$$

$$=\delta U\frac{dU}{d\varphi}-a\nu\left\{\frac{2}{\delta}(U-U_0)+\frac{1}{6\delta}KU\right\}.$$

Eliminating $a\nu$ by

$$a\nu=\frac{\delta^2}{K}\cdot\frac{dU}{d\varphi}$$

the last term in (11) becomes

$$-\frac{2\delta}{KU}U(U-U_0)\frac{dU}{d\varphi}-\frac{\delta}{6}U\frac{dU}{d\varphi}.$$

Further the following notations will be used

$$(11a)\qquad \frac{U}{U_0}=X \qquad\qquad \frac{KU}{10U_0}=Y.$$

After division by $\delta U_0 \frac{dU}{d\varphi}$ the equation (11) becomes

$$2\left\{\frac{7}{10} + \frac{1}{12}\frac{dY}{dX} + \frac{367}{630}(X-1) + \frac{71}{1512}(X-1)\frac{dY}{dX} + \frac{71}{1512}Y + \frac{100}{9072}Y\frac{dY}{dX}\right\} +$$

$$+ \frac{2}{\delta}\frac{d\delta}{dX}\left\{-\frac{3}{10}(X-1) + \frac{1}{12}Y - \frac{263}{1260}(X-1)^2 + \frac{71}{1512}(X-1)Y + \frac{100}{18144}Y^2\right\} -$$

$$- X\left\{\frac{7}{10} + \frac{1}{12}\frac{dY}{dX}\right\} + \frac{1}{\delta}\cdot\frac{d\delta}{dX}\left\{\frac{3}{10}X(X-1) - \frac{1}{12}XY\right\} =$$

$$= X - \frac{1}{5}\frac{X(X-1)}{Y} - \frac{1}{6}X .$$

Now $\frac{1}{\delta}\frac{d\delta}{dX}$ has to be expressed by X, Y and known functions.

As

$$Y = \frac{\delta^2}{10a\nu}\cdot\frac{U}{U_0}\cdot\frac{dU}{d\varphi}$$

then

$$\frac{dY}{dX} = \frac{2\delta}{10a\nu}\cdot\frac{d\delta}{dX}\cdot\frac{U}{U_0}\cdot\frac{dU}{d\varphi} + \frac{\delta^2}{10a\nu U_0}\left[\left(\frac{dU}{d\varphi}\right)^2 + U\frac{d^2U}{d\varphi^2}\right] : \frac{dU}{U_0 d\varphi}$$

and

$$\frac{1}{Y}\cdot\frac{dY}{dX} = \frac{2}{\delta}\cdot\frac{d\delta}{dX} + \frac{1}{X} + \frac{U_0\frac{d^2U}{d\varphi^2}}{\left(\frac{dU}{d\varphi}\right)^2} .$$

Using $$U = 2U_\infty(\sin\varphi + \sin\gamma)$$

one gets $$\frac{U_0\frac{d^2U}{d\varphi^2}}{\left(\frac{dU}{d\varphi}\right)^2} = -\frac{U_0}{2U_\infty}\cdot\frac{\sin\varphi}{1-\sin^2\varphi} .$$

Introducing a constant c by

$$4\,U_\infty = c\,U_0$$

and expressing $\sin\varphi$ by

$$\sin\varphi = \frac{U}{2U_\infty} - \sin\gamma = \frac{U_0}{2U_\infty}X - \sin\gamma = \frac{2}{c}X - \sin\gamma$$

there is

$$\frac{U_0 \frac{d^2U}{d\varphi^2}}{\left(\frac{dU}{d\varphi}\right)^2} = \frac{4X - 2c\sin\gamma}{4X^2 - 4Xc\sin\gamma - c^2\cos^2\gamma}\,.$$

Thus

$$\frac{2}{\delta}\frac{d\delta}{dX} = \frac{1}{Y}\cdot\frac{dY}{dX} - \frac{1}{X} - \frac{4X - 2c\sin\gamma}{4X^2 - 4Xc\sin\gamma - c^2\cos^2\gamma}$$

$$= \frac{1}{Y}\frac{dY}{dX} - \frac{2X^2 - 1.5Xc\sin\gamma - 0.25c^2\cos^2\gamma}{X(X^2 - Xc\sin\gamma - 0.25c^2\cos^2\gamma)}\,.$$

Further Y' is written for $\frac{dY}{dX}$ and the momentum equation becomes after multiplication by $\frac{Y}{2}$,

$$Y'\Big\{\frac{100}{9072}Y^2 + \frac{71}{1512}(X-1)Y + \frac{1}{12}Y - \frac{3}{20}(X-1) + \frac{1}{24}Y - \frac{263}{2520}(X-1)^2 + \frac{71}{3024}(X-1)Y +$$

$$+\frac{100}{36288}Y^2 - \frac{1}{24}XY + \frac{3}{40}X(X-1) - \frac{1}{48}XY\Big\} +$$

$$+\frac{7}{10}Y + \frac{367}{630}(X-1)Y + \frac{71}{1512}Y^2 - \frac{7}{20}XY - \frac{5}{12}XY + \frac{1}{10}X(X-1) -$$

$$-\frac{Y(2X^2 - 1.5Xc\sin\gamma - 0.25c^2\cos^2\gamma)}{2X(X^2 - Xc\sin\gamma - 0.25c^2\cos^2\gamma)}\Big\{-\frac{3}{10}(X-1) + \frac{1}{12}Y - \frac{263}{1260}(X-1)^2 + \frac{71}{1512}(X-1)Y + \frac{100}{18144}Y^2 + \frac{3}{20}X(X-1) - \frac{1}{24}XY\Big\} = 0.$$

This equation is now multiplied by 72.576 and solved with respect to Y'

$$Y' = \{Y(2X^2 - 1.5Xc\sin\gamma - 0.25c^2\cos^2\gamma)(0.2Y^2 + 0.192XY + 1.32Y - 2.1312X^2 - 1.1808X + 3.312) +$$

$$+X(X^2 - Xc\sin\gamma - 0.25c^2\cos^2\gamma)(-3.408Y^2 + 13.3632XY - 8.5248Y + 7.2576X - 7.2576X^2)\} : \tag{12}$$

$$: \{X(X - 0.5c\sin\gamma + 0.5c)(X - 0.5c\sin\gamma - 0.5)(Y - 1.2X + 1.2)(Y + 1.776X + 2.76)\}.$$

10. X varies on the upper side of the cylinder from zero at the stagnation point to $0.5c(1+\sin\gamma)$ at $\varphi = \frac{\pi}{2}$ and decreases then to zero at the rear stagnation point. On the lower side X decreases from zero at the forward stagnation point to $0.5c(-1+\sin\gamma)$ at $\varphi = -\frac{\pi}{2}$ and increases again to zero at the rear stagnation point.

As Y is the product of a positive quantity with $U\frac{dU}{d\varphi}$ then it is zero at the forward stagnation point, assumes positive values for increasing φ and decreases to zero at $\varphi = \frac{\pi}{2}$, then it is negative for larger values of φ and becomes zero again at the rear stagnation point. On the lower side of the cylinder Y is zero at the forward stagnation point, assumes further negative values until $\varphi = -\frac{\pi}{2}$, where it is zero again, then becomes positive and decreases to zero at the rear stagnation point.

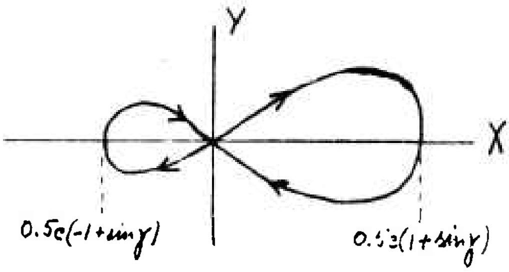


Fig.2

Thus the solution (Fig.2) of the differential equation must start from (0,0), proceed through positive values of X and Y to $(0.5c(1+\sin\gamma),0)$ and return through negative values of Y to (0,0). The solution for the lower part of the cylinder starts from (0,0), proceeds through negative values of X and Y to $(0.5c(\sin\gamma - 1),0)$ and returns through positive values of Y to (0,0).

The question is whether for a given γ there is a solution satisfying these conditions for every given value of c or only for certain values of c or for none.

One should expect a solution for only one value of c because a circulation in the flow should be produced uniquely by a certain spin of the cylinder.

To find an answer to the question the singularities of the differential equation have to be investigated.

Y' becomes infinite on five straight lines given by the factors in the denominator of (12). (Fig.3.)

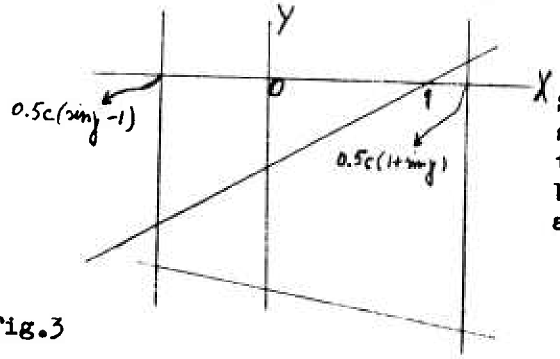


Fig.3

Suppose $0.5c(1+\sin\gamma) > 1$. The solution starting from (0,0) has to cross the line $Y = 1.2(X-1)$ between $X = 1$ and $X = 0.5c(1+\sin\gamma)$ and proceed then to $(0.5c(1+\sin\gamma),0)$.

On the line $Y = 1.2(X-1)$ there is a singular point between 1 and $0.5c(1+\sin\gamma)$ where the numerator in (12) is zero. This can be shown by substituting in the numerator $Y = 1.2(X-1)$. It then becomes

$$(X-1)\Big[(-1.93536X^2 - 0.48384X + 2.4192)(2X^2 - 1.5Xc\sin\gamma - 0.25c^2\cos^2\gamma) +$$
$$+ X(3.87072X - 5.32224)(X^2 - Xc\sin\gamma - 0.25c^2\cos^2\gamma)\Big].$$

The expression in the square brackets is equal to

$$-1.45152(1 - c\sin\gamma - 0.25\,c^2\cos^2\gamma)$$

for $X = 1$. The expression is positive, as according to the assumption

$$1 - 0.5c\sin\gamma < 0.5c$$

and the expression in question can be written

$$-1.45152\left[(1 - 0.5c\sin\gamma)^2 - (0.5c)^2\right].$$

For $X = 0.5c(1+\sin\gamma)$ the expression becomes

$$0.25c^2(1+\sin\gamma)(2.4192 - 0.48384X - 1.93536X^2)$$

and this is negative for $X > 1$.

Thus the numerator in (12) is zero somewhere between $X = 1$ and $X = 0.5c(1+\sin\gamma)$. The point, let us call it P, on the line $Y = 1.2(X-1)$ where the numerator is zero is a saddle point (Fig.4.) The derivative Y' is positive to the right of P above the line and to the left below the line, it is negative to the left of P above the line and to the right below the line. There are two integral curves which pass through the point P. Only one of them can possibly be the solution of the problem, that which passes through the point P from the left above the line to the right below the line. This curve has to pass through (0,0) to be a solution. This is

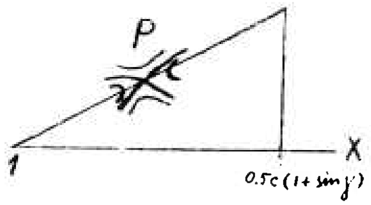


Fig.4

generally not the case. For $\gamma = \pi/2$ the computation showed that it did not pass through (0,0) for values of c slightly exceeding one (1.05 and 1.1). That suggests that there is no solution for $\gamma = \pi/2$ and c different from one. It has not yet been investigated whether there exist values of γ and c with $0.5c(1+\sin\gamma) \neq 1$ for which an integral curve through the singular point P reaches the origin. If such values do not exist then the conclusion is that

$$c = \frac{2}{1+\sin\gamma} .$$

It may be mentioned that in the case $0.5c(1+\sin\gamma) > 1$ there are two more singular points on the line $Y = 1.2(X-1)$. One is a spiral point at $X = 1$ and the other a saddle point between $X = 0$ and $X = 1$.

For $\gamma = \pi/2$, $c = 1$ an integral curve was computed numerically. A saddle point is in this case at (0.5, − 0.6) and the curve passing through this point passed through (0,0) and (1,0). The solution is shown in Fig.5.

From the found solution the boundary layer profiles were computed and are shown in Fig. 6.

In Fig. 7 the values of the form parameter K are plotted against the angle φ and in Fig. 8 the shearing stress τ times $\frac{\sqrt{Re}}{8\sqrt{2}\,\rho\, U_\infty^2}$ against φ. As

$$\tau = \mu \frac{\partial u}{\partial \eta}\Big|_{\eta=0}$$

then from (9),(10) and (11a) with $Re = \frac{U_\infty 2a}{\nu}$ follows

$$\frac{\tau\sqrt{Re}}{16\sqrt{2}\cdot\frac{1}{2}\rho\, U_\infty^2} = \left[2(X-1) + \frac{1}{6}KX\right]\left(\frac{dX}{d\varphi}/K\right)^{\frac{1}{2}} .$$

This is a dimensionless quantity.

Numerical integration gives

$$\frac{\sqrt{Re}}{8\sqrt{2}\,\rho\, U_\infty^2}\int_0^{2\pi} \tau\, d\varphi = -1.00$$

and thus the shearing stress on the surface of the cylinder is

$$\int_0^{2\pi} \tau \, a \, d\varphi = -a \, \ell \times 11.3 \, \rho \, U_\infty^2 \, Re^{-\frac{1}{2}}$$

and the power P necessary to rotate the cylinder so that $U_0 = 4U_\infty$ is

$$P = 45.2 \, a \ell \rho \, U_\infty^3 \, Re^{-\frac{1}{2}}.$$

11. The computation suggests that there exist integral curves for $0.5c(1 + \sin\gamma) = 1$ only. This equation gives the connection between the spin of the cylinder and the circulation, but it will be valid for the upper side of the cylinder only giving the stagnation points and therewith the region of attached flow on the upper side of the cylinder in dependence on the velocity of rotation of the cylinder.

What happens on the lower side of the cylinder?

If $U_0 = 0$ then there is separation on the upper as well as on the lower side of the cylinder between 80° and 90° from the stagnation point. If U_0 has a positive value the point of separation on the upper side will be moved farther from the forward stagnation point and the separation point on the lower side will be nearer to the forward stagnation point than in the case $U_0 = 0$, because on the lower side the wall is moving in a direction opposite to the flow.

Thus there will be a wake on the lower side, the limits of which are a point between $\varphi = -\gamma$ and $\varphi = -\frac{\pi}{2}$ and the rear stagnation point $\varphi = \pi + \gamma$.

This has to be taken into account in solving (12) for negative values of X. The solution is limited by the line $Y = 1.2(X - 1)$. As long as $|Y| < 1.2|X - 1|$ a family of integral curves exist which pass through (0,0) and $0.5c(\sin\gamma - 1)$. But for larger values of $|Y|$ there are curves starting from (0,0) which cross the line $Y = 1.2(X - 1)$ and remain on the lower side of it (Fig.9). These do not reach the point $0.5c(\sin\gamma - 1)$ and correspond to a separated flow. To calculate such a flow the pressure distribution from the forward stagnation point to the point of separation must be known. Of course one can get an approximate solution taking on the lower side of the cylinder as well $U = 2U_\infty(\sin\varphi + \sin\gamma)$ as if the flow was attached.

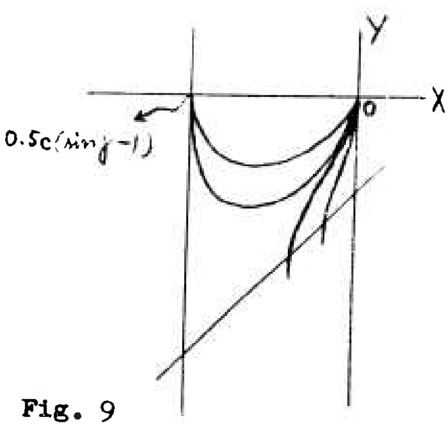


Fig. 9

For the calculation of the Magnus force the pressure in the wake must be known.

LIST OF REFERENCES

[1.] J. M. Burgers, Verslag van de Gewone vergaderingen der Wisen-Naturkundige Afdeeling 29(1921) p.952.

[2.] S. Goldstein, Modern Developments of Fluid Dynamics, p.152.

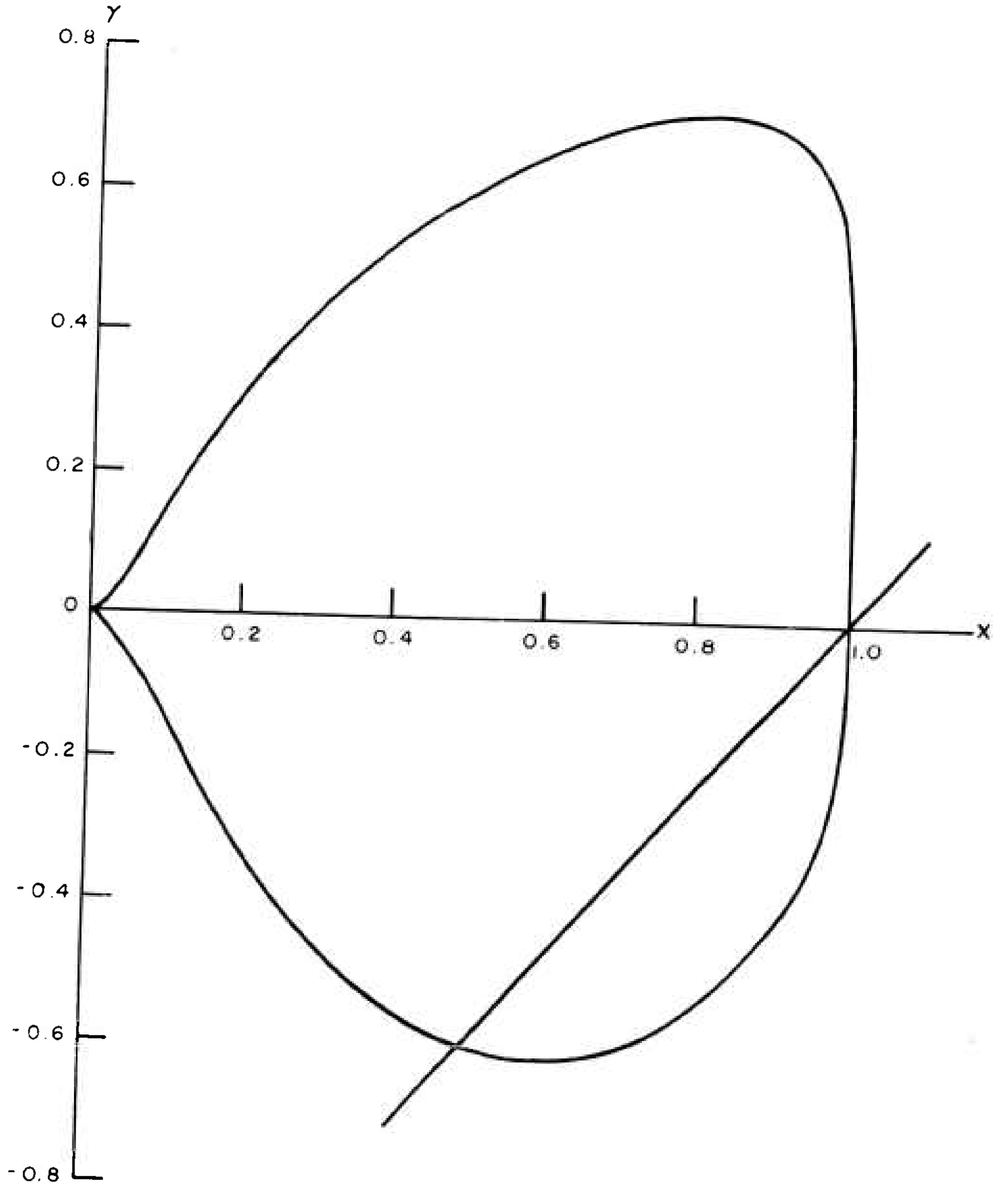


FIG. 5 SOLUTION IN THE CASE OF ONE STAGNATION POINT

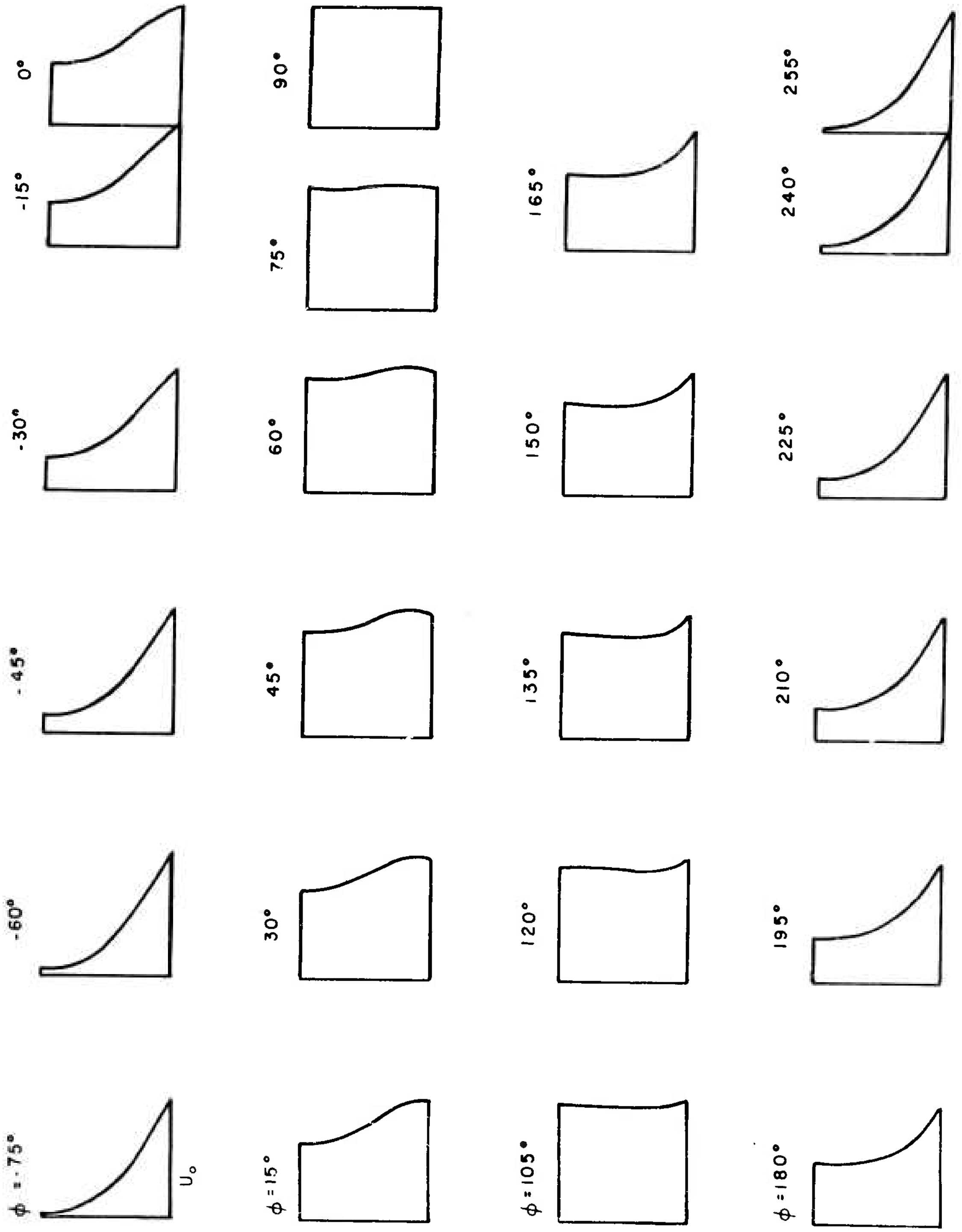


FIG. 6 VELOCITY PROFILES

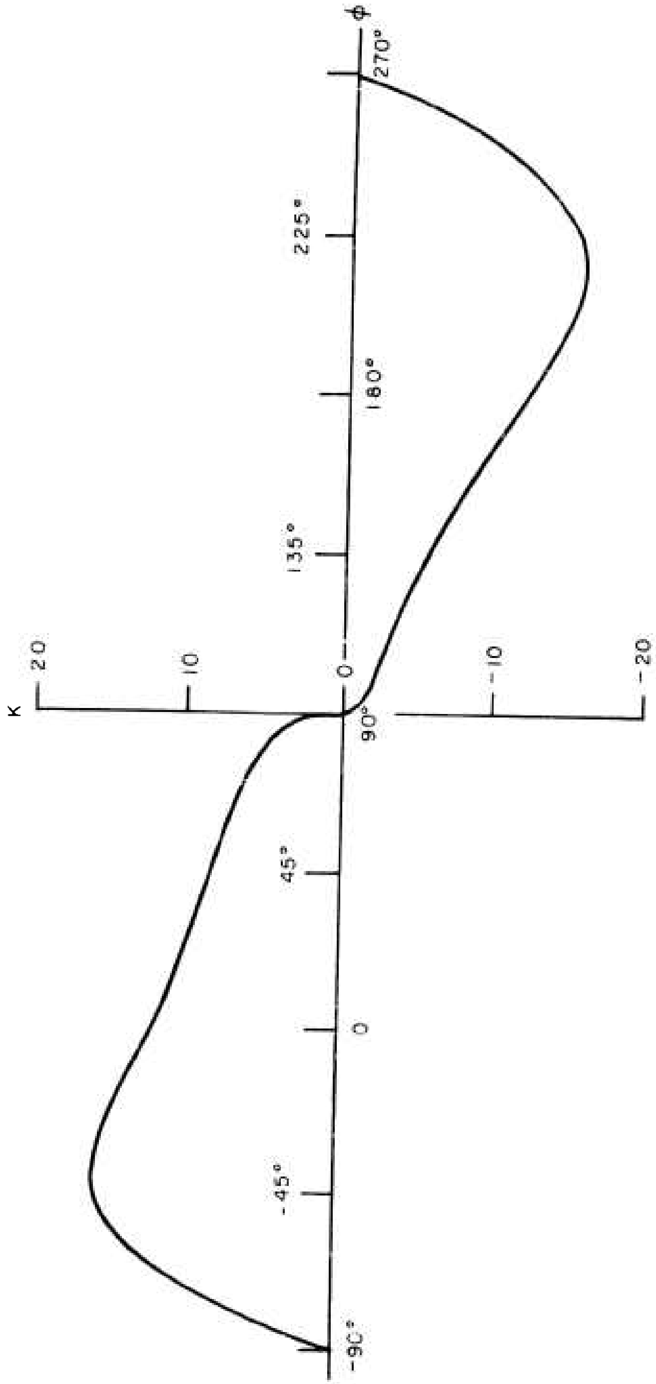
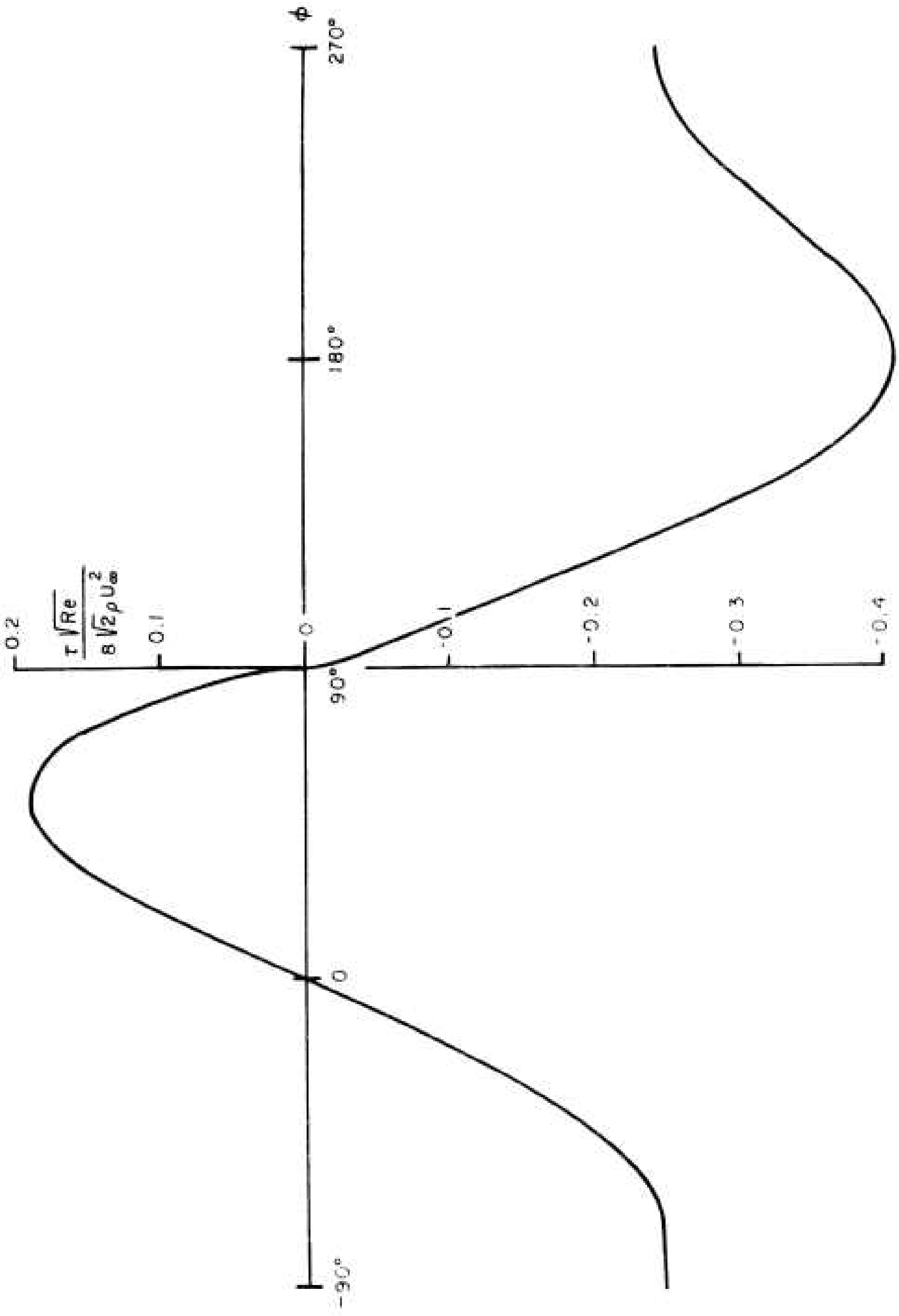


FIG. 7 THE FORM PARAMETER K



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